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A Study on Difference Equations and their Applications in Different Kind of Industries

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Abstract: *The study of topics in discrete mathematics usually includes the study of algorithms, their implementations, and efficiencies. Discrete mathematics is the mathematical language of computer science, and as such, its importance has increased dramatically in recent decades. Mathematical computations frequently are based on equations that allow us to compute the value of a function recursively from a given set of values. Such an equation is called a "difference equation" or "recurrence equation". These equations occur in numerous settings and forms, both in mathematics itself and in its applications to statistics, computing, electrical circuit analysis, dynamical systems, economics, biology, etc.*

I. INTRODUCTION

Difference equation is the branch of mathematics dealing with objects that can assume only distinct, separated values. The term "discrete mathematics" is therefore used in contrast with "continuous mathematics," which is the branch of mathematics dealing with objects that can vary smoothly (and which includes, for example, calculus). Whereas discrete objects can often be characterized by integers, continuous objects require real numbers.

Example 1

In 1626, Peter Minuit purchased Manhattan Island for goods worth \$24. If the \$24 could have been invested at an annual interest rate of 7% compounded quarterly, what would it have been worth in 2004?

Solution

Let y(t) be the value of the investment after t quarters of a year. Then y(0)=24. Since the interest rate is 1.75% per quarter, y(t) satisfies the difference equation

$$y(t+1) = y(t) + 0.0175y(t) = 1.0175y(t) \quad t = 0,1,2,\cdots$$

Computing *y* recursively, we have

$$y(1) = 24(1.0175)$$

$$y(2) = 24(1.0175)^2$$

$$\vdots$$

$$y(t) = 24(1.0175)^t$$

In 378 years, or 1512 quarters, the value of the investment is

$y(1512)=24(1.0175)^{1512} \approx 5.9x10^{12}$ (about 5.9 trillion dollars!)

Example 2

It is observed that the decrease in the mass of a radioactive substance over a fixed time period is proportional to the mass that was present at the beginning of the time period. If the half life of radium is 1600 years, find a formula for its mass as a function of time.

Solution

Let *m(t)* represent the mass of the radium after *t* years. Then

$m(t+1)-m(t)=-km(t)$

where *k* is a positive constant. Then

$m(t+1)=(1-k)m(t), \quad t=0,1,2\ldots$

Using iteration as in the previous example, we find

$$m(t) = m(0)(1-k)^t$$

Since the half life is 1600 years

$$m(1600) = m(0)(1-k)^{1600} = \frac{1}{2}m(0) \Rightarrow 1-k = \left(\frac{1}{2}\right)^{\frac{1}{1600}}$$

so

$$m(t) = m(0)\left(\frac{1}{2}\right)^{\frac{t}{1600}}$$

(In physics, this problem is usually solved using an integral of a differential equation. The solution presented here is somewhat shorter and employs only elementary algebra)

The starting equations in the above examples were all difference equations. By definition, an equation which expresses the value a_n of a sequence $\{a_n\}$ as a function of the term a_{n-1} is called *first-order difference equation.* If we can find a function *f* such that $a_n=f(n)$, $n=1, 2, 3\ldots,$ then we will have solved the difference equation.

Both examples had a general difference equation of the form

$$x_{n+1} = Ax_n, \qquad n = 0,1,2,\cdots$$

From the above examples, it is obvious that if x_0 is the initial value, then

$$x_n = A^n x_0$$

Given the constants *A* and *B*, a difference equation of the form

$$x_{n+1} = Ax_n + B, \qquad n = 0,1,2,\cdots$$

is called a *first-order linear difference equation*. Note that the linear difference equation reduces to the difference equation if B=0. In order to solve this type of equation, we write

$$\begin{aligned} x_n &= Ax_{n-1} + B = A(Ax_{n-2} + B) + B = A^2 x_{n-2} + B(A+1) \\ &= A^2(Ax_{n-3} + B) + B(A+1) = A^3 x_{n-3} + B(A^2 + A + 1) \\ &\vdots \\ &= A^n x_0 + B(A^{n-1} + A^{n-2} + \cdots A^2 + A + 1). \end{aligned}$$

Note that if A=1, this gives us

$$x_n = x_0 + nB, \qquad n = 0,1,2,\cdots$$

as the solution of the difference equation $x_{n+1}=x_n+B$. If $A\neq 1$, we know that

$$A^{n-1} + A^{n-2} + \cdots + A^2 + A + 1 = \frac{1-A^n}{1-A}$$

Hence

$$x_n = A^n x_0 + B\left(\frac{1-A^n}{1-A}\right)$$

is the solution of the first-order linear difference equation $x_{n+1}=Ax_n+B$, when $A\neq 1$.

Example (Newton's Law of Cooling)

If T_0 represents the initial temperature of the object, *S* the constant temperature of the surrounding environment and T_n the temperature of the object after *n* units of time, then the change in the temperature over one unit of time is given by

$$T_{n+1} - T_n = k(T_n - S)$$

Where *k* is a constant which depends upon the object. This difference equation is known as *Newton's law of cooling*. Suppose a cup of tea, initially at a temperature of *82 C*, is placed in a room which is held at a constant temperature of *26 C*. Moreover, suppose that after one minute the tea has cooled to *80 C*. What will the temperature be after *20* minutes?

Solution

If we let T_n be the temperature of the tea after *n* minutes and we let *S* be the temperature of the room, then we have $T_0=82$, $T_1=80$, and $S=26$. Newton's law of cooling states that

$$T_{n+1} - T_n = k(T_n - 26), \quad n = 0,1,2,\cdots$$

Where *k* is the constant which we have to determine. To do so, we make use of the information given about the change in the temperature of the tea during the first minute. When $n=0$, we have

$$T_1 - T_0 = k(T_0 - 26) \quad or \quad 80 - 82 = k(82-26) \quad so \quad k = -0.035$$

Thus, the original equation reduces to

$$T_{n+1} - T_n = -0.035(T_n - 26) = -0.035T_n + 0.91$$

$$so$$

$$T_{n+1} = T_n - 0.035T_n + 0.91 = 0.965T_n + 0.91$$

Now the last equation is in the standard form of a first-order linear difference equation, so from the solution (according to the method of solving presented earlier)

$$T_n = 0.965^n(82) + 0.91\left(\frac{1-0.965^n}{1-0.965}\right) = 0.965^n(82) + 26(1-(0.965)^n) =$$

$$= 26 + 56(0.965)^n$$

In particular,

$$T_{20} = 26 + 56(0.965)^{20} \approx 53.5$$

So after *20* minutes the tea has cooled to just below *54 C*.

An equation of the type

$$au_{n+2} + bu_{n+1} + cu_n = f(n)$$

Where *a, b* and *c* are constants and *f(n)* a given function is called a *second order constant coefficient difference equation*. The equation is said to be homogeneous if *f(n)=0* and inhomogeneous if *f(n)≠0*.

A. Homogeneous Second Order Difference Equations

We can have an idea on how these equations can be solved by looking first at first-order equation with constant coefficient

$$u_{n+1} - cu_n = 0$$

which has the solution

$$u_n = Ac^n$$

where A is any constant. With this in view, we attempt to find solutions of

$$au_{n+2} + bu_{n+1} + cu_n = 0$$

in the form

$$u_n = p^n$$

Where p is a constant. Thus

$$au_{n+2} + bu_{n+1} + cu_n = ap^{n+2} + bp^{n+1} + cp^n = (ap^2 + bp + c)p^n = 0$$

for all n. The solution $p=0$ leads to the trivial solution $u_n=0$. We are interested in the roots of the equation in the brackets which is called the characteristic equation.

According to the nature of the roots of the characteristic equation, we can have different solutions.

Case 1: Distinct roots

The general solution of the second-order difference equation for distinct roots p_1 and p_2 of $ap^2+bp+c=0$ is

$$u_n = Ap_1^n + Bp_2^n$$

for any constant *A* and *B*.

Example

Find the general solution of

$$u_{n+2} - u_{n+1} - 6u_n = 0$$

Solution

The characteristic equation is $p^2-p-6=0$ or $(p-3)(p+2)=0$. This means that the roots are $p_1=3$ and $p_2=-2$. Hence the general solution is

$$u_n = A\cdot 3^n + B\cdot(-2)^n$$

Example

Find the solution of

$$u_{n+2} + 2u_{n+1} - 3u_n = 0$$

that satisfies $u_0=1$, $u_1=2$.

Solution

The characteristic equation is

$$p^2 + 2p - 3 = 0 \quad or \quad (p+3)(p-1) = 0$$

The roots are $p_1=-3$ and $p_2=1$. Hence the general solution is

$$u_n = A(-3)^n + B\cdot 1^n = A(-3)^n + B$$

Using the initial conditions

$$u_0 = 1 = A + B, \quad u_1 = 2 = -3A + B$$

Hence $A=-1/4$ and $B=5/4$. The required solution is $u_n = -\frac{1}{4}(-3)^n + \frac{5}{4}$

Case 2: Equal roots ($p_1=p_2=p$)

The general solution of $au_{n+2} + bu_{n+1} + cu_n = 0$ is

$$u_n = (A + Bn)p^n$$

(Please note the difference in the form of the second-order difference equation)

Case 3: Complex roots

The general complex solution of $au_{n+2} + bu_{n+1} + cu_n = 0$, where $b^2<4ac$, is

$$u_n = A(\alpha + i\beta)^n + B(\alpha - i\beta)^n$$

This solution can be written in other form by writing the complex numbers in polar coordinates *(r,θ)*. The connection between the two forms of writing a complex number is

$$r=\sqrt{\alpha^2+\beta^2},\quad \cos\theta=\frac{\alpha}{r},\quad \sin\theta=\frac{\beta}{r},\quad \tan\theta=\frac{\beta}{\alpha}$$

So the general solution of a second-order homogeneous difference equation with complex roots can be also written as

$$u_n=r^n\left(C\cos\theta n+D\sin\theta n\right)$$

Example

Obtain the general solution of

$$u_{n+2}+u_n=0$$

Solution

The characteristic equation is $p^2+1=0$, giving roots $p_1=i$ and $p_2=-i$. Therefore

$$u_n=Ai^n+B(-i)^n$$

In polar form, $r=1$ and

$$\sin\theta=1\Rightarrow\theta=k\frac{\pi}{2}$$

where k is an integer number. So, the solution is

$$u_n=C\cos n\frac{\pi}{2}+D\sin n\frac{\pi}{2}$$

B. Inhomogeneous Second-Order Difference Equations

The general inhomogeneous equation is

$$au_{n+2}+bu_{n+1}+cu_n=f(n)$$

We write $u_n=v_n+q_n$, where v_n is the general solution of the corresponding homogeneous equation. Substitute this form of u_n into the general equation

$$a(v_{n+2}+q_{n+2})+b(v_{n+1}+q_{n+1})+c(v_n+q_n)=f(n)$$

or

$$\left(av_{n+2}+bv_{n+1}+cv_n\right)+\left(aq_{n+2}+bq_{n+1}+cq_n\right)=f(n)$$

Since v_n satisfies the homogeneous equation, it follows that

$$aq_{n+2}+bq_{n+1}+cq_n=f(n),$$

Which means that q_n must be a particular solution of the inhomogeneous equation. As in the theory of differential equations, v_n is known as the complementary function.

We construct particular solutions by appropriate choices of functions usually containing adjustable parameters which are suggested by the form of the function *f(n)*. In the following table you will find the suggested forms of particular solutions

$k\ (a\ const)$	$C;\ or\ Cn,\ if\ C\ fails;\ Cn^2,\ if\ C\ and\ Cn\ fail;\ etc$
k^n	$Ck^n;\ or\ Cnk^n,\ if\ Ck^n\ fails;\ etc$
n	C_0+C_1n
$n^p\ (p\ \text{int}\ eger)$	$C_0+C_1n+\cdots C_pn^p$
$\sin kn\ or\ \cos kn$	$C_1\cos kn+C_2\sin kn$

Example

Obtain the general solution of

$$u_{n+2}-u_{n+1}-6u_n=4$$

Solution

From the example solved earlier, the complementary solution is

$$v_n=3^nA+(-2)^nB$$

For the particular solution, we try $q_n=C$ (see the table above). Then

$$q_{n+2}-q_{n+1}-6q_n-4=C-C-6C-4=-6C-4=0\Rightarrow C=-\frac{2}{3}$$

Hence $q_n=-2/3$, and the general solution is

$$u_n=3^nA+(-2)^nB-\frac{2}{3}$$

Example

Obtain the general solution of

$$u_{n+2}+2u_{n+1}-3u_n=4$$

Solution

From a previous example, we know that the complementary solution is

$$v_n = (-3)^n A + B$$

In this case we expect a particular solution of the form $q_n=C$. However, if we introduce this form into the general solution, we would obtain that the left-hand side of the equation is identical zero. That means that this choice of the particular solution fails, so we are choosing $q_n=Cn$. Then

$$q_{n+2} + 2q_{n+1} - 3q_n - 4 = C(n+2) + 2C(n+1) - 3C - 4 = 4C - 4 = 0 \Rightarrow C = 1$$

Hence the general solution is

$$u_n = (-3)^n A + B + n$$

II. REFERENCES

[1] Juang, J.-N., Applied System Identification, Prentice-Hall, Englewood Cliffs, NJ, 1994, pp. 175-252.

[2] Juang, J.-N., Phan, M.Q., Horta, L.G., and Longman, R.W., Identification of Observer/Kalman Filter Markov Parameters: Theory and Experiments, Journal of Guidance, Control and Dynamics, Vol. 16, No. 2, 1993, pp. 320-329.

[3] S. J. Julier,J. K. Uhlmann, and H. F. Durrant-Whyte A New Approach for Filtering Nonlinear Systems , Proceedings of the 1995 American Control Conference, Seattle, Washington, Pages:1628-1632,1995.

[4] Mohinder S Grewal, Angus P, Andrews Kalman filtering theory and practice using matlab , John wiley & sons 2001. [8] Shaked. U and C. E. de Souza, Robust minimum variance filtering, IEEE Trans. Signal. Process, vol. 43, no. 11, pp. 2474– 2483, Nov. 2002.

[5] Simon.D and Chia.T.L, Kalman Filtering with State Equality Constraints, IEEE Trans. On Aerospace and Electronic Systems, 38, 1 (Jan. 2002), 128-136

[6] Yang.C, Bakich.M, and Blasch.E, Nonlinear Constrained Tracking of Targets on Roads, Proc. Of the 8th International Conf. on Information Fusion, Philadelphia, PA, July 2005