Solution of Convex Feasibility Problems with Quasi Non-Expansivity

Smit Yadav
(Lecturer-in-Maths, GPES, Lisana, Rewari, Haryana, India)

Abstract: *This paper provides a projection method for convex feasibility problems that is known to converge only weakly. Exploiting the property concerning the intersection of a family of convex closed sets, we present a condition that makes them strongly convergent without additional assumptions.*

I. INTRODUCTION

The convex feasibility problem consists in finding a point in the intersection of a convex set. Initially this problem arose in the constraint optimization problems for trying to guess an initial feasible point that is a point which satisfies the constraints. Often these constraints are defined by linear inequalities and so the feasible set is the intersection of a number of half spaces. Later, it was proved that the convex feasibility problem have great utility and broad applicability in many areas, spreading on modern mathematical and physical science to economics and even medical practice, like statistics, electron microscopy signal processing and the like. A complete and exhaustive study on algorithms for solving convex feasibility problem, including comments about their applications and an excellent bibliography was given by H.H. Bausche and J. M. Borwein.

The projection algorithms it seems to be the common way for solving this problem. The idea is to use the projection of the current iterate onto certain set from the intersection family and so to yielding a sequence of points that is supposed to converge to a solution. Here this paper presents simple condition that ensures the strong convergence of the sequence generated by the projection algorithm.

II. PRELIMINARIES

Let H be a real Hilbert space with scalar product $\langle .,. \rangle$, norm $\| . \|$, and distance d. Let $T : D \subset H \to H$ be a nonlinear mapping, and let F(T) denotes the set of fixed points of T in D. In the following we will assume that $F(T) \neq \phi$. The mapping T is said to be quasi-non-expansive if $\| Tx - x \| \le \| x - x^* \|, \forall x \in D, x^* \in F(T).$ Let $d(x, E)$ denotes the distances between a point $x \in H$ and a set $E \subset H$ that is $d(x, E) = \inf_{y \in E} \| x - y \|$.

Definition: A mapping $T : C \to C$ is said to be quasi-non expansive if $F(T) \neq \phi$ and $\phi(p, Tx) \le \phi(p, x), \ \forall x \in C, p \in F(T)$.

Definition: A Banach space E is said to be strictly convex if $\| x + y \| / 2 < 1$ for all $x, y \in U = \{ z \in E : \| z \| = 1 \}$ with $x \neq y$.

Results

Lemma: Let $M_i \subset H(1, 2, ..., m)$ be a family of convex sets such that $\bigcap M_i$ is non-empty and bounded and let $\{x_k\}$ be a sequence of H such that $d(x_k, M_i) \to 0 as k \to \infty$ for each i. Then $d(x_k, M_i \to 0), as k \to \infty$.

Proof: Let us assume that $o \in \bigcap M_i$. Then there exists a closed ball $D(o, r) = \{ x \in H : \| x \| \le r \} \subset \bigcap M_i$. Let $\in$ be a given real number, $0 < \in < 1$ and let C be a constant such that $\| x \| \le C - 1$ for all $x \in \bigcap M_i$, which is possible because $\bigcap M_i$ is bounded.

Since $d(x_k, M_i) \to 0$ as $k \to \infty$, for each index I, there exists a sequence $\{ y_k^{(i)} \}_{k \in N} \subset M_i$ such that $\| y_k^{(i)} - x^k \| \to 0 as k \to \infty$. Let

$$Z_k = \left(1-\frac{C}{\in}\right)\left(y_k^{(i)} - x_k\right), k = 0,1,\ldots \quad (1.1)$$

There exist a number $k_i(\in)$ such that if $k \geq k_i(\in)$ then $\| y_k^{(i)} \| \leq \frac{r}{|1-\frac{C}{\in}|}$ and so $\| Z_k \| \leq r$, that is $Z_k \in \bigcap M_i$.

From (1.1), we have

$\left(1-\frac{C}{\in}\right)x_k = \frac{\in}{C}Z_k + \left(1-\frac{\in}{C}\right)y_k^{(i)}$, and for $k \geq k_i(\in)$ we have $\left(1-\frac{\in}{C}\right)x_k \in M_i$, because $y_k^{(i)}, Z_k \in M_i$ are convex. Now let $k_0(\in)$ it follows that $\left(1-\frac{\in}{C}\right)x_k \in \bigcap M_i$ and

$$d\left(x_k, \bigcap M_i\right) \leq \| x_k - \left(1-\frac{\in}{C}\right)x_k \| = \frac{\in}{C-\in}\left\| \left(1-\frac{\in}{C}\right)x_k \right\| < \in$$

Hence the proof of the lemma.

Theorem: Let $M_i\left(i = 1, 2, \ldots, m\right)$ be a family of closed and convex sets of H such that $\bigcap M_i$ is nonempty and bounded. Then the sequence $\langle x_k \rangle$ is given as $X_k = T_\lambda^k x_0$ converges strongly to a point of $\bigcap M_i$ for all $x_0 \in H$.

Proof: Since $F\left(T_\lambda\right) = \bigcap M_i$ is a closed set, it suffices to show that T_λ is quasi-non-expansive on H and that $d\left(x_k, \bigcap M_i\right) \to 0$ as $k \to 0$.

Let $x \in H$ and $y \in M_i$. Since $P\left(x, i_x\right)$ is the projection of x onto M_{i_x} and $y \in M_{i_x}$, we have

$$\langle Tx - y, x - Tx \rangle = \langle P\left(x, i_x\right) - y, x - P\left(x, i_x\right) \rangle \geq 0$$

and

$$\begin{aligned} \| T_\lambda x - y \|^2 &= \| x - y \|^2 - 2\lambda \langle x - y, x - Tx \rangle + \lambda^2 \| x - Tx \|^2 \\ &= \| x - y \|^2 - \lambda\left(2-\lambda\right)\| x - Tx \|^2 - 2\lambda \langle Tx - y, x - Tx \rangle \quad (1.2) \\ &\leq \| x - y \|^2 - \lambda\left(2-\lambda\right)\| x - Tx \|^2 \end{aligned}$$

So we have

$$\| T_\lambda x - y \| \leq \| x - y \|, \forall x \in H, y \in \bigcap M_i \quad (1.3)$$

and T_λ is quasi-non-expansive on H.

Now, since $x_{k+1} = T_\lambda x_k$, from (1.3) it follows that the sequence $\left\{\| x_k - y \|\right\}$ is monotone decreasing and bounded therefore $\| x_k - y \| \to \delta_y ask \to \infty$, for each $y \in \bigcap M_i$ .From (1.2) we have

$$\| x_k - Tx_k \|^2 \leq \frac{1}{\lambda\left(2-\lambda\right)}\left(\| x_k - y \|^2 - \| x_{k+1} - y \|^2\right)$$

And hence $\| x_k - Tx_k \| \to 0$ as $k \to \infty$. But $\| x - P\left(x_k, i\right) \| \leq \| x - Tx \|$ for each i. Therefore $d\left(x_k, M_i\right) = \| x_k - P\left(x_k, i\right) \| \to 0$ as $k \to \infty$.

III. APPLICATIONS

This method can have multiple applications in practice. For example, in computerized tomography the speed in which the images are obtained is of special interest, due of the necessity of real time working, hence the convergence rate of the algorithm is an important matter. In such a case this method can constitute a routine of the computerized tomography system. This routine involves the reconstruction of a planar image in real time and then the three dimensional one. Moreover, this method allows to study the convergence speed of the projection algorithms.

IV. REFERENCES

[1] Bauschke, H.H., Borwein, J.M., On projection algorithms for solving convex feasibility problems, Review, vol.38, No.3.

[2] Gubin, L.G., Polyac, B.T., Raik, E. V., The method of projections for finding the common point of convex point of convex sets, USSR Comput. Math. Pp 1-24.

[3] Diaz, J.B., Metcalf, F.T., On the set of sub-sequential limit points of successive approximations, Trans. Amer. Math. pp. 459-485.

[4] 4 .A George, G.W. Stewart and R. Voigt Special volume of Linear Algebra and its Applications on parallel computing , Linear Algebra Appl. 77.

[5] 5 .G.T. Herman, Image Reconstruction from Projections: The fundamentals of Computerized Tomography, Academic, New York.

[6] 6. G.T. Herman, H. Levkowitz, H.K Tuy and S. McCormick, Multilevel image reconstruction, in Multiresolution Image Processing and Analysis, Springer-Verlag 121-125.

[7] 7. Combettes, P.L., Pennanen, T., General Mann iterates for constructing fixed points in Hilbert spaces, J. Math. Anal.,pp. 521-536.

[8] 8. H.H., Kurk, Bauschke, S.G. The method of reflection-projection for convex feasibility problems with an obtuse cone, Technical report, Oakland Univ., Rochester MI.

[9] 9. Bregman, L.M., The method successive projection for finding a common point of convex sets, Soviet Math. pp. 688-692.

[10] 10. Agmon, S., The relaxation method for linear inequalities, Canad. J. Math., Vol. 6, pp. 382-392.

[11] 11. Laura Maruster and Stefan Maruster, On convex feasibility problems, pp. 1-4.