Inequality and Maximal-Minimal solutions for Non-linear Differential and Integral Equations

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Abstract: In the present research work we study some basic outcomes related to strict and non-strict non-linear differential and integral inequalities and existence of maximal and minimal solutions are proved for a Non-linear differential equation.

Index Terms: Non-linear differential equations, Strict and Non-Strict Inequalities, Existence theorem, Maximal and Minimal Solutions

1. INTRODUCTION

Let R be real line which is connected set, i.e., which cannot be written in the form of union of two separated sets. (Two sets A and B are said to be separated if $\bar{A}\bigcap B=\phi \; and \; A\bigcap \bar{B}=\phi$)Given a bounded interval $I=[t_0,t_0+b]$ in R for some fixed $t_0,b\in R \;\; with \;\; b>0$.

Consider the initial value problems for non-linear differential equations (NDE)

$$\frac{d}{dt}\left[x(t)-f(t,x(\alpha(t)))\right]=g(t,x(\alpha(t))),\ t\in I$$

$$x(t_0)=x_0\in R \quad and \quad \alpha(t)\in R \qquad (1.1)$$

Where $f,g:I\times R\to R$ are continuous real valued functions defined on I.

By a solution of the (1.1), we mean a function $x\in C(I,R)$ such that

i) the function $t\to x-f(t,x(\alpha(t)))$ is continuous for each $x\in R$ and $\alpha(t)$ be any scalar valued function.

ii) x satisfies the equation (1.1).

The importance of the investigations of the work of non-linear differential equations lies in the fact that they include various dynamic systems as special cases.[6,15,16] The consideration of non-linear differential equations is implicit in the work of Krasnoselskii[13] and extensively treated in the various research papers on non-linear differential equations with different perturbations. see Krasnoselskii[13] and references therein. This class of differential and integral equations includes the perturbation of original differential and integral equations in different ways [7,8,9,10,11,12].

In this paper, we initiate the basic theory of non-linear differential equation's of mixed inequalities and existence theorem. Our claim is that, the outcomes of this paper are of basic level and significant contribution to the theory of non-linear ordinary differential equations.

II. STRICT AND NON-STRICT INEQUALITIES

We need frequently the following hypothesis in what follows:

(A_0) The function $x\to x-f(t,x(\alpha(t)))$ is increasing in R for all $t\in I$.

We begin by proving the basic results dealing with non-linear differential inequalities.

Theorem 2.1: Assume that the hypothesis (A_0) holds. Suppose that there exist $y,z\in C(I,R)$

suchthat $$\frac{d}{dt}\left[y(t)-f(t,y(\alpha(t)))\right]\le g\left[t,y(\alpha(t))\right],\ t\in I \qquad (2.1)$$

and $$\frac{d}{dt}\left[z(t)-f(t,z(\alpha(t)))\right]\ge g\left[t,z(\alpha(t))\right],\ t\in I \qquad (2.2)$$

If one of the inequalities (2.1) and (2.2) is strict and

$$y(t_0)<z(t_0) \qquad (2.3)$$

then $y(t)<z(t)$ (2.4)

for all $t\in I$.

Proof:-Suppose that the inequality (2.4) is false, then the set Z defined by

$$Z = \{t \in I : y(t) \geq z(t)\} \qquad (2.5)$$

is non empty.

Denote $t_1 = \inf Z$ without loss of generality, we may assume that

$$y(t_1) = z(t_1) \quad and \quad y(t) < z(t) \quad for\ all\ t < t_1.$$

Assume that $\frac{d}{dt}\left[z(t) - f(t, z(\alpha(t)))\right] > g(t, z(\alpha(t))) \quad for\ \ all\ \ t \in I.$

Denote

$$Y(t) = \left[y(t) - f(t, y(\alpha(t)))\right] \quad and$$

$$Z(t) = \left[z(t) - f(t, z(\alpha(t)))\right] \quad for\ \ all\ \ t \in I.$$

Now continuity of y and z together with (2.3) implies that there exists a $t_1 > t_0$ *such that*

$$y(t_1) = z(t_1) \ \ and \ \ y(t) < z(t) \qquad (2.6)$$

For all $t_0 \leq t \leq t_1$.

As (A_0) holds,it follows from (2.5) that

$$Y(t_1) = y(t_1) - f(t_1, y(\alpha(t_1)))$$

$$= z(t_1) - f(t_1, z(\alpha(t_1))) = Z(t_1).$$

and

$$Y(t) = y(t) - f(t, y(\alpha(t)))$$

$$< z(t) - f(t, z(\alpha(t)))$$

$$\Rightarrow Y(t) < Z(t) \qquad (2.7) \quad for\ all\ t_0 \leq t < t_1.$$

From the above relation (2.7) , we obtain

$$Y(t_1 + h) = Z(t_1 + h) \ \ and \ \ Y(t_1) < Z(t_1)$$

$$\Rightarrow -Y(t_1) > -Z(t_1)$$

$$\therefore Y(t_1 + h) - Y(t_1) > Z(t_1 + h) - Z(t_1)$$

Dividing both sides by $h \neq 0$

$$\frac{Y(t_1 + h) - Y(t_1)}{h} > \frac{Z(t_1 + h) - Z(t_1)}{h}$$

For small h < 0.

Taking the limit as $h \to 0$, we obtain

$$Y'(t_1) \geq Z'(t_1) \qquad (2.8)$$

Hence from inequality (2.7) and (2.8) , we get

$$g(t_1, y(\alpha(t_1))) \geq Y'(t_1) \geq Z'(t_1) > g(t_1, z(\alpha(t_1)))$$

Which is a contradiction o our assumption that $y(t) \geq z(t)$.

Hence $\qquad y(t) < z(t) \quad for\ \ all\ \ t \in I.$

In the next theorem we discuss the non-strict inequality for the non-linear differential equation (1.1) on I in which one sided Lipschitz condition used.

Theorem 2.2:- Assume that the hypothesis of theorem 2.1 hold. Suppose also that there exists a real number L>0 such that

$$g(t, y(\alpha(t))) - g(t, z(\alpha(t))) \leq L \sup_{t_0 < r < t}\left[y(r) - f(r, y(\alpha(r))) - (z(r) - f(r, z(\alpha(r))))\right] \qquad (2.9)$$

Whenever $y(r) \geq z(r),\ t_0 \leq r < t$. then

$$y(t_0) \leq z(t_0) \qquad (2.10)$$

$$\Rightarrow y(t) \leq z(t) \qquad (2.11)$$

$$for\ all\ t \in I.$$

Proof:- Let $\in > 0$ be given and let L>0 be any given real number.Set define

$$z_\in(t) - f(t, z_\in(\alpha(t))) = z(t) - f(t, x(\alpha(t))) + \in e^{2P(t-t_0)} \qquad (2.12)$$

$$so\ that$$

$$z_\in(t) - f(t, z_\in(\alpha(t))) > z(t) - f(t, x(\alpha(t)))$$

We define

$$Z_\in(t) = z_\in(t) - f(t, z_\in(\alpha(t)))\ and\ Z(t) = z(t) - f(t, z(\alpha(t)))\quad for\ \ all\ \ t \in I.$$

Now using inequality (2.9), we have

$$g(t, z_\in(\alpha(t))) - g(t, z(\alpha(t))) \leq L \underset{t_0 \leq r < t}{Sup}\left[Z_\in(r) - Z(r)\right]$$

$$= L \in e^{2L(t-t_0)}$$

Now ,

$$Z'_\in(t) = Z(t) + 2L \in e^{2L(t-t_0)}$$

$$\geq g(t, z(t)) + 2L \in e^{2L(t-t_0)}$$

$$\geq g(t, z_\in(t)) + 2L \in e^{2L(t-t_0)} - L \in e^{2L(t-t_0)}$$

$$\Rightarrow Z'_\in(t) \geq g(t, z_\in(t)) \quad for\ \ all\ \ t \in I.$$

Also we have $Z_\in(t_0) > Z(t_0) \geq Y(t_0).\quad for\ \ all\ \ t \in I.$ Now using theorem 2.1 with $z = z_\in$, to give $Y(t) < Z_\in(t)\ \ for\ all\ \ t \in I.$

On taking $\in \to 0$ $in\ \ the\ \ above\ \ inequalty,we\ \ get\ Y(t) \leq Z(t)$

Which is further in view of hypothesis (A_0) implies that (2.11) holds on I. Hence the proof.

III. EXISTENCE RESULT

In this article we prove an existence result for the non-linear differential equation (1.1) on a closed and bounded interval $I = [t_0, t_0 + b]$ under the mixed Lipschitz and CompactnessConditions on the non- linearity involved in it.

We use the non-linear differential equation (1.1) in the space C(I,R) of continuous real valued Functions defined on $[t_0, t_0 + b]$

In C(I,R) we define a supremum norm $\|\cdot\|$ as $\|x\| = \underset{t \in I}{sup}|x(t)|$. Clearly C(I,R) is a separable Banach space with respect to the above supremum norm. We prove The existence of solutions for the non-linear differential equation (1.1) via the following fixed point theorem in the Banach spaces.[4]

Theorem 3.1 Suppose that S is closed,convex and bounded subset of the separable Banach space Eand let $A : E \to E$ and $B : S \to E$ be two operators such that

a) A is non-linear contraction

b) B is compact and continuous, and

c) $x = Ax + By\quad for\ \ all\ \ y \in S \Rightarrow x \in S$.

Then the operator equation $Ax + By = x$ has a solution in S.

We consider the following hypothesis in what follows.

(A_1)There exists a constant L>0 such that $|f(t,x) - f(t,y)| < \dfrac{L|x-y|}{M + |x-y|}$,

for all $t \in I\ and\ \ p, q \in R. moreover\ \ L \leq M$.

(A_2)There exists a continuous function $h : I \to R$ such that $|g(t,p)| \leq h(t)\ \ t \in I,\ \ for\ \ all\ \ p \in R.$

To prove the theorem the following lemma is useful which is discussed in sequel.

Lemma 3.1 Assume that hypothesis (A_0) holds. Then for any continuous function $h: I \to R$, the function $x \in C(I,R)$ is a solution of non-linear differential equation

$$\frac{d}{dt}\left[x(t) - f(t, x(\alpha(t)))\right] = h(t), \quad for \quad all \quad t \in I,$$

$$x(0) = x_0 \in R \qquad (3.1)$$

If and only if x satisfies the non-linear differential equation

$$x(t) = x_0 - f(t_0, x(\alpha(t_0))) + f(t, x(\alpha(t))) + \int_{t_0}^{t} h(s)ds \qquad (3.2)$$

Proof :- Let $h \in C(I,R)$

We first assume that x satisfy the (3.1) then by definition $x(t) - f(t, x(\alpha(t)))$ is continuous on the interval $I = [t_0, t_0 + b)$ and so it is differentiable there, as a result $\frac{d}{dt}\left[x(t) - f(t, x(\alpha(t)))\right]$ is integrable on I.

Integrating (3.1) from t_0 to t, we have

$$\int_{t_0}^{t} \frac{d}{dt}\left[x(t) - f(t, x(\alpha(t)))\right] dt = \int_{t_0}^{t} h(t)dt$$

$$\left[x(t) - f(t, x(\alpha(t)))\right]_{t_0}^{t} = \int_{t_0}^{t} h(s)ds$$

$$x(t_0) = x_0$$

$$i.e., \left[x(t) - f(t, x(\alpha(t)))\right] = \left[x(t_0) - f(t_0, x(\alpha(t_0)))\right] + \int_{t_0}^{t} h(s)ds, \ t \in I$$

$$\therefore x(t) = x(t_0) - f(t_0, x(\alpha(t_0))) + f(t, x(\alpha(t))) + \int_{t_0}^{t} h(s)ds \quad t \in I.$$

Conversely suppose that x satisfies

$$x(t) = x(t_0) - f(t_0, x(\alpha(t_0))) + f(t, x(\alpha(t))) + \int_{t_0}^{t} h(s)ds \quad t \in I.$$

Differentiating above equation we get $\frac{d}{dt}\left[x(t) - f(t, x(\alpha(t)))\right] = h(t), \ t \in I$

Now substituting $t = t_0$ in (3.2) ,we get

$$x(t_0) = x_0 - f(t_0, x(\alpha(t_0))) + f(t_0, x(\alpha(t_0)))$$

$$\therefore x(t_0) - f(t_0, x(\alpha(t_0))) = x_0 - f(t_0, x(\alpha(t_0)))$$

Since the mapping $x \mapsto x - f(t, x)$ is an increasing in R for all $I \in R$. Also the mapping $x \mapsto x - f(t_0, x)$ is one one in R. This proves $x(t_0) = x_0$ .This completes the lemma.

Now we are going to discuss the following existence theorem for the Non-linear differential equation (1.1) on the interval I.

Theorem 3.2 : Assume that the hypothesis (A_0) – (A_2) hold. Then the non-linear differential equation (1.1) has a solution defined on I.

Proof: Let set E=C(I,R) and define a subset S of E defined by

$$S = \{x \in E \,\|\, \|x\| \le N\} \qquad (3.3)$$

Where $N = |x_0 - f(t_0, x(\alpha(t_0)))| + L + F_0 + \|h\|$

$$F_0 > 0 \ \ such \ that \ F_0 = \sup_{t \in I} |f(t, x(\alpha(t)))|$$

Clearly S is a closed , convex and bounded subset of the Banach space E.
Now using the hypothesis (A_0) and (A_2) and application of lemma 3.1, we can easily show that the non-linear differential equation (1.1) is equivalent to the non-linear integral equation

$$x(t) = x_0 - f(t_0, x(\alpha(t_0))) + f(t, x(\alpha(t))) + \int_{t_0}^{t} g(s, x(\alpha(s)))ds \qquad (3.4)$$

for $t \in J$.

We define two operators $A: E \to E$ and $B: S \to E$ by

$$Ax(t) = f(t, x(\alpha(t))) \ \ , t \in J \qquad (3.5)$$

And

$$Bx(t) = x_0 - f(t_0, x(\alpha(t_0))) + \int_{t_0}^{t} g(s, x(\alpha(s)))ds \ \ , t \in I \qquad (3.6)$$

Then the integral equation (3.5) is transformed into an operator equation as

$$Ax(t) + Bx(t) = x(t) \ \ , t \in J \qquad (3.7)$$

Our aim is to show that the operators A and B satisfy all the conditions of theorem (3.1).
i.e., we first show that A is a Lipschitz operator on E with the Lipschitz constant L.
Let p,q be any two members in E, then by hypothesis (A_1)

$$\begin{aligned} |A(x(t)) - A(y(t))| &= |f(t, x(\alpha(t))) - f(t, y(\alpha(t)))| \\ &\le \frac{L|x(\alpha(t)) - y(\alpha(t))|}{M + |x(\alpha(t)) - y(\alpha(t))|} \\ &\le \frac{L\|x-y\|}{M + \|x-y\|} \qquad for \ \ all \ \ \alpha(t) \in R, where\ t \in I. \end{aligned}$$

This shows that A is a non-linear contraction E with D-function ψ defined by $\psi(r) = \dfrac{Lr}{M+r}$.

Now we have to show the second condition of theorem (3.1)i.e., B is compact and continuous operator on S into E.First we prove, B is continuous on S.

Let $\{p_n\}$ be a sequence in S converging to a point $x \in S$. Then by dominated convergence theorem for integration, we obtain

$$\begin{aligned} \lim_{n\to\infty} Bp_n(t) &= \lim_{n\to\infty}\left[x_0 - f(t_0, x(\alpha(t_0))) + \int_{t_0}^{t} g(s, p_n(\alpha(s)))ds \right] \\ &= \lim_{n\to\infty} x_0 - \lim_{n\to\infty} f(t_0, x(\alpha(t_0))) + \lim_{n\to\infty}\int_{t_0}^{t} g(s, p_n(\alpha(s)))ds \\ &= x_0 - f(t_0, x(\alpha(t_0))) + \int_{t_0}^{t}\left[\lim_{n\to\infty} g(s, p_n(\alpha(s)))\right]ds \\ &= x_0 - f(t_0, x(\alpha(t_0))) + \int_{t_0}^{t} g(s, x(\alpha(s)))ds \\ &= Bx(t) \ \ , \quad for \ \ t \in I. \end{aligned}$$

Moreover, it can be shown as below that $\{Bp_n\}$ is an equicontinuous sequence of functions in X. Now, following the arguments similar to that given in Granaset.al[16], it is proved that B is a continuous operator on S.

Now we have to show B is compact operator on S.
To prove this it is sufficient to show that B(S) is a uniformly bounded and equicontinuousset in E.

Let $x \in S$ be arbitrary. Then by hypothesis (A_2).

$$|Bx(t)| \le |x_0 - f(t_0, x(\alpha(t_0)))| + \int_{t_0}^{t} |g(s, x(\alpha(s)))| ds$$
$$\le |x_0 - f(t_0, x(\alpha(t_0)))| + \int_{t_0}^{t} h(s) ds$$
$$\le |x_0 - f(t_0, x(\alpha(t_0)))| + \|h\| a \quad for\ all\ \ t \in I$$

Taking supremum over t,

$$\underset{t}{Sup} |Bx(t)| \le \underset{t}{Sup} \{ |x_0 - f(t_0, x(\alpha(t_0)))| + \|h\| a \}$$
$$\|Bx\| \le |x_0 - f(t_0, x(\alpha(t_0)))| + \|h\| a \ \ , \ for\ all\ x \in S.$$

This shows that B is uniformly bounded on S. Again let $t_1, t_2 \in I$ .then for any $x \in S,$ we have

$$|Bx(t_1) - Bx(t_2)| = \left| \int_{t_0}^{t_1} g(s, x(\alpha(s))) ds - \int_{t_0}^{t_2} g(s, x(\alpha(s))) ds \right|$$
$$\le \left| \int_{t_2}^{t_1} g(s, x(\alpha(s))) ds \right|$$
$$\le |p(t_1) - p(t_2)|$$

Where $p(t) = \int_{t_0}^{t} h(s) ds.$

Since the function p is continuous on compact I, it is uniformly continuous.

Hence, for $\in > 0$, there exists a $\delta > 0$ such that $|t_1 - t_2| < \delta \Rightarrow |Bx(t_1) - Bx(t_2)| < \in$

For all $t_1, t_2 \in I$ and for all $x \in S.$

This shows that B(S) is an equicontinuous set in E.

Now being uniformly bounded and equicontinuous set in E, so it is compact by Arzela-Ascoli theorem. This proves, B is a continuous and compact operator on S.

Now we have to show that $p = Ap + Bq \quad for\ all\ y \in S \Rightarrow x \in S\ is\ satisfied.$

Let $p \in E$ and $q \in S$ be arbitrary such that $x = Ax + By$.

Then, by assumption (A_1) , we have

$$|x(t)| = |Ax(t) + By(t)|$$
$$\le |Ax(t)| + |By(t)|$$
$$\le |x_0 - f(t_0, x(\alpha(t_0)))| + |f(t, x(\alpha(t)))| + \int_{t_0}^{t} |g(s, y(\alpha(s)))| ds$$
$$\le |x_0 - f(t_0, x(\alpha(t_0)))| + \left[|f(t, x(\alpha(t))) - f(t,0)| + |f(t,0)| \right] + \int_{t_0}^{t} |g(s, y(\alpha(s)))| ds$$
$$\le |x_0 - f(t_0, x(\alpha(t_0)))| + L + F_0 + \int_{t_0}^{t} h(s) ds$$

Taking supremum over t,

$$\|x\| \le |x_0 - f(t_0, x(\alpha(t_0)))| + L + F_0 + \|h\| b$$

Thus all the conditions of theorem (3.1) are satisfied and hence the operator equation $Ax + By = x$ has a solution in S.As a result, the non-linear differential equation (1.1) has a solution defined on I. This completes the proof.

IV. MAXIMAL AND MINIMAL SOLUTIONS

Under this section we shall discuss the existence of maximal and minimal solutions for the Non-linear differential equation (1.1) on $I=[t_0,t_0+b]$.

Definition : - A solution of the Non-linear differential equation (1.1) is said to be Maximal if for any other solution x to the Non-linear differential equation (1.1) one has $x(t)\le r(t),\ for\ all\ t\in I$ .Again , a solution ρ of the Non-linear differential equation (1.1)is said to be minimal if $\rho(t)\le x(t),\ for\ all\ t\in I,$ where x is any solution of the Non-linear differential equation (1.1) existing on I.

We study the case of Maximal solutions only, as the case of minimal solution is similar and can be proved with the suitable and appropriate modifications.

Given a arbitrary small real number $\in>0$, consider the following IVP of Non-linear differential equation

(1.1)
$$\frac{d}{dt}\left[x(t)-f(t,x(\alpha(t)))\right]=g(t,x(\alpha(t))+\in,t\in I \qquad (4.1)$$
$$x(t_0)=x_0+\in$$

Where $f,g\in C(I\times R,R)$.

An existence theorem for the Non-linear differential equation (1.1) can be stated as follows:

Theorem 4.1 Assume that the hypothesis (A_0)-(A_2) hold. Then for every small number $\in>0$, the Non-linear differential equation (1.1) has a solution defined on I.

Theorem 4.2 Assume that the hypotheses (A_0)-(A_2) hold. Further $L\le M$, then the Non-linear differential equation (1.1) has a maximal solution defined on J.

Proof:-Let $\{p_n\}_0^\infty$ be a decreasing sequence of positive real numbers such that $\lim_{n\to\infty} p_n=0.$ Then for any solution v of the NDE(1.1) , by theorem 2.1 , one has

$$v(t)<r(t,p_n) \qquad (4.2)$$

for all $t\in I$ and $n\in N\cup\{0\}$, where $r(t,p_n)$ is a solution of the NDE,

$$\frac{d}{dt}\left[x(t)-f(t,x(\alpha(t)))\right]=g(t,x(\alpha(t)))+p_n,\ \ t\in I$$
$$x(t_0)=x_0+p_n \qquad (4.3)$$

defined on J.

Since by theorem 3.1 and 3.2 , $\{r(t,p_n)\}$ is a decreasing sequence of positive real numbers, the limit exists. We show that the convergence in (4.6) is uniform in I. To finish, it is sufficient to prove that the sequence $\{r(t,p_n)\}$ is equicontinuous in $C(I,R)$.

Let $t_1,t_2\in I$ be arbitrary. Then,

$$\begin{aligned}
&|r(t_1,p_n)-r(t_2,p_n)|\\
&\le|f(t_1,r(t_1,p_n))-f(t_2,r(t_2,p_n))|+\left|\int_{t_0}^{t_1}g(s,r_{p_n}(s))ds-\int_{t_0}^{t_2}g(s,r_{p_n}(s))ds\right|+\left|\int_{t_0}^{t_1}p_n ds-\int_{t_0}^{t_2}p_n ds\right|\\
&=|f(t_1,r(t_1,p_n))-f(t_2,r(t_2,p_n))|+\left|\int_{t_1}^{t_2}g(s,r_{p_n}(s))ds\right|+\left|\int_{t_1}^{t_2}p_n ds\right|\\
&\le|f(t_1,r(t_1,p_n))-f(t_2,r(t_2,p_n))|+\left|\int_{t_1}^{t_2}h(s)ds\right|+\left|\int_{t_1}^{t_2}p_n ds\right|\\
&=|f(t_1,r(t_1,p_n))-f(t_2,r(t_2,p_n))|+\left|\int_{t_1}^{t_2}h(s)ds\right|+|t_1-t_2|p_n\\
&=|f(t_1,r(t_1,p_n))-f(t_2,r(t_2,p_n))|+|p(t_1)-p(t_2)|+|t_1-t_2|p_n \qquad (4.5)
\end{aligned}$$

Where $p(t)=\int_{t_0}^{t_1} h(s)ds.$

Since *f is continuous on compact set* $I\times[-N,N]$, they are uniformly continuous there. Hence,

$$\left| f(t_1, r(t_1, p_n) - f(t_2, r(t_2, p_n)\right| \to 0 \text{ as } t_1 \to t_2$$

uniformly for all $n\in N$. Similarly the function k is continuous on compact set J, it is uniformly continuous and hence

$$|k(t_1)-k(t_2)| \to 0 \text{ as } t_1 \to t_2 \text{ uniformly for all } t_1, t_2 \in I.$$

Therefore, from the above inequality (4.5) , it follows that

$$|r(t_1, p_n) - r(t_2, p_n)| \to 0 \text{ as } t_1 \to t_2$$

uniformly for all $n\in N$. Therefore,

$$r(t,p_n) \to r(t) \text{ as } n \to \infty, \text{ for all } t \in I.$$

Next, we show that the function r(t) is a solution of the NDE (3.1) defined on I. Now, since $r(t,p_n)$ is a solution of the NDE(4.5), we have

$$r(t,p_n) = x_0 + p_n + f(t, r(t,p_n)) + \int_{t_0}^{t_1} g(s, r_{p_n}(s))ds \qquad (4.6)$$

for all $t\in I.$ Taking the limit as $n\to\infty$ in the above equation (4.6) yields

$$r(t) = x_0 - m(t_0, x_0) + f(t, r(\alpha(t))) + \int_{t_0}^{t_1} g(s, r(\alpha(s)))ds$$

for all $t\in I.$ Thus the function r is a solution of the NDE(1.1) on I. Finally from the inequality (4.4) it follows that

$$v(t) \le r(t)$$

for all $t\in I.$ Hence the NDE(1.1) has a maximal solution on I. This completes the proof.

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