On the Additive and Multiplicative Structures of Idempotent Semirings

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Abstract: In this paper, we study the classes of Multiplicatively subidempotent semirings and extended this concept and studied idempotent semiring. We also studied inter relations between different semirings.

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I. INTRODUCTION

The word idempotent signifies the study of semirings in which the addition operation is idempotent a + a = a. The best known example for idempotent semiring is the max-plus semiring. Interest has been shown in such structures arose in late 1950s through the observation that certain problems of discrete optimization could be linearized over suitable idempotent semirings. Recently the subject has established connections with discrete event systems automata theory, non-expansive mappings, optimization theory.

Idempotent semiring is a fundamental structure that has widespread applications in Computer Science. Idempotent semiring is a ring with additive idempotent. Recently modal operators of idempotent semirings are introduced in order to model the properties of programs and transition systems more suitably and to link algebraic and relational formalisms with dynamic and temporal logics. Section one deals with classes of multiplicatively subidempotent semiring and in section two we generalized subidempotent to idempotent semirings and proved few results on idempotent semiring.

II. PRELIMINARIES

A triple (S, +, •) is a semiring if S is a non-empty set and “+, •” are binary operations on S satisfying that (i) The additive reduct (S, +) and the multiplicative reduct (S, •) are semigroups and (ii) a (x + c) = ax + ac and (x + c) a = xa + ca, for all a, x, c in S. A semiring (S, +, •) is denoted by S.

In a semiring S, an element a is Multiplicatively Subidempotent if a + a^2 = a. A semiring S is Multiplicatively Subidempotent if and only if each of its elements is Multiplicatively Subidempotent. A semiring S is said to be mono if a + b = ab for all a, b in S.A semigroup (S, •) is a band if a^2 = a for all a in S. A semigroup (S, +) is a band if a + a = a for all a in S. A semigroup (S, •) is left (right) singular if ax = a (xa = x) for all a, x in S. A semigroup (S, +) is left (right) singular if a + x = a (ax = x) for all a, x in S.A semiring S is idempotent if a + a = a and a^2 = a for all a in S. In an additive idempotent semiring S if for each a in S there exists an element x in S such that a + axa = axa, then S is a K–regular semiring. An element a of a semigroup (S, •) is left (right) regular if axa = ax and axa = xa for all a, x in S. An element a in a semigroup (S, +) is periodic if ma = na where m and n are positive integers. A semigroup (S, +) is periodic if every one of its elements is periodic. A semigroup (S, +) is lateral zero if axc = x.

III. CLASSES OF MULTIPLICATIVELY SUBIDEMPOTENT SEMIRING

Proposition 1: Suppose S is multiplicatively subidempotent and mono semiring. Then a^{2n} = a^2 for n > 1 and a^{2n+1} = a for all a in S and n ≥ 1.

Proof: Since S is multiplicatively subidempotent, a + a^2 = a.
Also S is mono semiring then a + a = aa for all a in S.
Using mono semiring in above we get a.a^2 = a implies a^3 = a → (1)
⇒ a^2 = a^3
Which implies a^5 = a^3 = a
This again leads to a^6 = a^4 = a^2
Therefore a^{2n} = a^2 for n > 1 and a^{2n+1} = a for n ≥ 1.
Theorem 1: If \( S \) is a multiplicatively subidempotent semiring and \((S, \cdot)\) is right singular semigroup, then \((S, +)\) is a band.

Proof: By hypothesis \((S, \cdot)\) is right singular then \(xa = a\) for all \(a, x\) in \(S\).
Also \(a + a^2 = a\) for all \(a\) in \(S\) that implies \(xa + xa^2 = xa\).

\[\Rightarrow a = a + a \quad \text{for all} \quad a \in S\]
Therefore \((S, +)\) is a band.

Proposition 2: Assume that \( S \) is multiplicatively subidempotent and monosemiring. Then

(a) \((S, \cdot)\) is a left regular semigroup, if \((S, +)\) is commutative.

(b) \(axa = a + axa\).

(c) \((S, +)\) is periodic.

(d) Square of every element in \( S \) is additively idempotent.

Proof:

(a) Since \( S \) is multiplicatively subidempotent \(a + a^2 = a\) \(\Rightarrow a + a^2 + x = a + x\).
By using \((S, +)\) commutative we get \(a + x + a^2 = a + x\).
Which leads to \(ax + a^2 = ax \Rightarrow (x + a) = ax \Rightarrow axa = ax\).
Thus \((S, \cdot)\) is a left regular semigroup.

(b) Given that \( S \) is monosemiring, \(a + x = ax\) \(\Rightarrow (1)\).
\[\Rightarrow a^2 + xa = axa\]
By addition of ‘\(a\)’ on both sides we obtain \(a + a^2 + xa = a + axa\).
\[\Rightarrow a + xa = a + axa\]
Using equation (1) in above we obtain \(axa = a + axa\).

(c) Since \( S \) is multiplicatively subidempotent semiring \(a + a^2 = a\) \(\Rightarrow (1)\).
And from monosemiring \(a + a = a.a\) \(\Rightarrow (2)\).
This implies \(a + a + a = a + a^2\).
\[\Rightarrow a + a + a = a \Rightarrow 3a = a\]
Hence \((S, +)\) is periodic.

(d) Since \( S \) is a monosemiring \(a + a = a^2\).
Also \( S \) is multiplicatively subidempotent semiring \(a + a^2 = a\).
\[\Rightarrow a + a + a^2 = a + a \Rightarrow a^2 + a^2 = a + a \Rightarrow a^2 + a^2 = a^2\]
Thus square of every element in \( S \) is additively idempotent.

Example 1: Consider a semiring \( S = \{a, b, x\} \) as an example for above proposition 2 which satisfies only \(axa = a + axa\).

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Example 2: \( S = \{a, y\} \) is an example for the above theorem which satisfies \((S, \cdot)\) left regularity, \((S, +)\) commutative and \(axa = a + axa\).

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Proposition 3: If \( S \) is a multiplicatively subidempotent semiring and \((S, +)\) is lateral zero, then \(x + ax^n = x\) for all \(a, x\) in \(S\).

Proof: Given that \( S \) is multiplicatively subidempotent, \(a + a^2 = a \Rightarrow axc + a^2 xc = axc\).
Since \((S, +)\) is lateral zero \(axc = x\) then above equation becomes \(x + ax = x\).
\[\Rightarrow x + a (x + x^2) = x\]
Which implies \(x + ax + ax^2 = x\).
Since \( x + ax = x \) then above equation becomes \( x + ax^2 = x \)
This again implies \( x + ax.x = x \)
The above can also be written as \( x + a(x + x^2) = x \)
\[ \Rightarrow x + ax^2 + ax^3 = x \Rightarrow x + ax^3 = x \]
Continuing in a similar manner we obtain \( x + ax^n = x \) for all \( a, x \) in \( S \)

IV. CLASSES OF IDEMPOTENT SEMIRING

**Theorem 2:** Suppose \( S \) is an idempotent semiring, \( \{a\} \) is additively completely regular element. Then \( a + ax = ax + a = a \) where \( x \) depends on the element \( a \).

**Proof:** By hypothesis \( \{a\} \) is additively completely regular element then there exists \( x \) such that \( a + x + a = a \) and \( a + x = x + a \)
Again let us consider \( a + x + a = a \) implies \( a + a + x = a \) \( \Rightarrow (1) \)
Since \( S \) is an idempotent semiring \( a + a = a \) and \( a^2 = a \)
Then equation (1) becomes \( a + x = a \Rightarrow a^2 + ax = a^2 \)
\[ \Rightarrow a + ax = a \]
Also \( x + a = a \) then proceeding as above we obtain \( ax + a = a \)
Therefore \( a + ax = ax + a = a \) where \( x \) depends on the element \( a \)

**Proposition 4:** If \( S \) is an idempotent semiring and \( (S, +) \) is cancellative, then \( (S, +) \) is commutative.

**Proof:**
(a) First let us consider the term \( ax + x^2 + a^2 + xa \)
\[ = (a + x) x + (a + x) a \]
The above equation can also be written as \( (a + x)(x + a) \)
\[ = a(x + a) + x(x + a) \]
This again equal to \( ax + a^2 + x^2 + xa \)
\[ \Rightarrow ax + x + a + xa = ax + a + x + xa \]
By using cancellation property we obtain \( x + a = a + x \)
Thus \( (S, +) \) is commutative
(b) By the definition of idempotent semiring we have \( (a + x)^2 = (a + x) \)
This implies \( (a + x)(a + x) = (a + x) \)
Which can be written as \( a(a + x) + x(a + x) = (a + x) \)
\[ \Rightarrow a^2 + ax + xa + x^2 = a + x \]
\[ \Rightarrow a + ax + xa + x = a + x \rightarrow (1) \]
Using left cancellation law in equation (1) we get \( ax + xa + x = x \)
Using right cancellation law in equation (1) we obtain \( a + ax + xa = a \)

**Theorem 3:** Suppose \( S \) is an Idempotent semiring. If \( a + ax + x = a \) and \( (S, \cdot) \) is left singular semigroup, then \( (S, +) \) is left singular semigroup.

**Proof:**
Given that \( S \) is an Idempotent semiring, \( a = a + a \)
By hypothesis we have \( a + ax + x = a \)
Since \( (S, \cdot) \) is left singular semigroup \( ax = a \) for all \( a, x \) in \( S \)
\[ \Rightarrow a + a + x = a \Rightarrow a + x = a \]
Thus \( (S, +) \) is left singular semigroup

**Theorem 4:** Let \( S \) be an Idempotent semiring. Then \( a + ax + x = ax \) if and only if \( a + ax = ax = ax + x \) for all \( a, x \) in \( S \).

**Proof:** By hypothesis \( S \) is idempotent, \( a = a + a \) and \( a^2 = a \) for all \( a \) in \( S \)
Also \( a + ax + x = ax \)
\[ \Rightarrow a^2 + a^2 x + ax = a^2 x \Rightarrow a + ax + ax = ax \Rightarrow a + ax = ax \rightarrow (1) \]
Again let us take \( a + ax + x = ax \)
\[ \Rightarrow a + ax + x = ax \Rightarrow ax + x = ax \rightarrow (2) \]
From equations (1) and (2) \( a + ax = ax = ax + x \)
To prove the converse part
Let us consider \( a + ax = ax = ax + x \)
\[ a + ax = ax \]
This implies \( a + ax + x = ax + x \)

\[ \Rightarrow a + ax + x = ax \]

V. REFERENCES