GRACEFULNESS IN PATH UNION OF VERTEX SWITCHING OF CYCLES

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Abstract

A graceful labeling of a graph $G$ with $q$ edges is an injection $f: V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ with the property that the resulting edge labels are also distinct, where an edge incident with the vertices $u$ and $v$ is assigned the label $|f(u) - f(v)|$. A graph which admits a graceful labeling is called a graceful graph. In this paper, we prove that the path union of vertex switching of cycle graphs is graceful.

Keywords: Graceful labeling, vertex switching, cycle, path union of graphs.

1. Introduction:

The most famous and challenging graph labeling method is the graceful labeling of graphs introduced by Rosa [9] in 1967. A graceful labeling of a graph $G$ with $q$ edges is an injection $f: V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ with the property that the resulting edge labels are also distinct, where an edge incident with the vertices $u$ and $v$ is assigned the label $|f(u) - f(v)|$. A graph which admits a graceful labeling is called a graceful graph. A variety of graphs and families of graphs are known to be graceful for the past five decades. Caterpillars are proved to be graceful by Rosa [9]. Morgan [8] has shown that all lobsters with perfect matchings are graceful. Hmiciar and Haviar [6] have shown that all trees of diameter five are graceful. Golomb [3] has proved that the complete bipartite graph $K_{m,n}$ is graceful. Rosa [9] showed that the $n$-cycle $C_n$ is graceful if and only if $n \equiv 0$ or $3 \pmod{4}$. Wheels $W_n = C_n + K_1$ is graceful [5]. Helms are shown to be graceful [1]. Kaneria et al. [7] have proved that the path union of complete bipartite graphs is graceful. Ghodasara et al. [4] have proved that the path union of vertex switching of cycle graphs is cordial. For an exhaustive survey on graceful graphs one may refer to the dynamic survey by Gallian [2]. In this paper, we prove that the path union of vertex switching of cycle graphs is graceful.

2. Main Result:

In this section we first recall the definition for cycle, vertex switching of a graph, path union of graphs. Then we prove that the path union of vertex switching of cycles is graceful.

Definition 1:

A sequence of vertices $[v_0, v_1, v_2, \ldots, v_n, v_0]$ is a cycle of length $(n+1)$ if $v_{i+1}v_i \in E, \ i = 0, 1, 2, 3 \ldots n$ and $v_nv_0 \in E$. A cycle of length $n$ is denoted by $C_n$.

Definition 2: [10]

A vertex switching of a graph $G$ is the graph $H$ which is obtained by taking a vertex $v$ of $G$, removing all the edges incident to $v$ and adding edges joining $v$ to every other vertex which are not adjacent to $v$ in $G$. We call $v$ as the switching vertex.
Definition 3:

Let $G$ be a graph and $G_1, G_2, \ldots, G_n$ ($n \geq 2$) be $n$ copies of graph $G$. Then the graph obtained by adding an edge from $G_i$ to $G_{i+1}$, $i = 1, 2, \ldots, n-1$ is called the path union of $G$.

3. Theorem

The path union of vertex switching of cycles is graceful.

Proof:

Let $G$ be a cycle $C_n$ with $n$ vertices that are denoted as $v_1, v_2, \ldots, v_n$ in the anticlockwise direction. Let $H$ be the vertex switching graph of the graph $G$ with $v_1$ as the switching vertex. The graphs $G$ and $H$ are shown in Figure 1.

![Figure 1: The graphs $G$ and $H$](image)

Consider $n$ copies of the graph $H$ and denote these copies as $H_1, H_2, \ldots, H_n$. The graphs $H_1, H_2, \ldots, H_n$ are shown in Figure 2. The first copy $H_1$ of $H$ is described as follows. Denote the switching vertex of first copy as $v^{1}_{1}$. Denote the remaining vertices in $H_1$ as $v^{1}_{2}, v^{1}_{3}, \ldots, v^{1}_{m}$ in the anticlockwise direction. In the second copy $H_2$ of $H$, the switching vertex is denoted as $v^{2}_{1}$ and the remaining vertices in $H_2$ are denoted as $v^{2}_{2}, v^{2}_{3}, \ldots, v^{2}_{m}$ in the anticlockwise direction. Finally the last copy $H_n$ of $H$ is described by denoting the switching vertex as $v^{n}_{1}$ and the remaining vertices of $H_n$ are denoted as $v^{n}_{2}, v^{n}_{3}, \ldots, v^{n}_{m}$ in the anticlockwise direction.

![Figure 2: Copies of the graph $H$](image)
Let $P$ be the graph obtained by adding an edge $e_i$ between the switching vertices $v_i^1$ and $v_i^{i+1}$ of the copies $H_i$ and $H_{i+1}$, $1 \leq i \leq (n - 1)$. The graph $P$ so obtained is called the path union of vertex switching of cycles and is shown in Figure 3

Figure 3: The graph $P$ which is the path union of vertex switching of cycles

Note that in $P$ the switching vertices are $v_i^i$ for $1 \leq i \leq n$ and the remaining vertices are $v_j^j$ for $(1 \leq i \leq n)$, $(2 \leq j \leq m)$. If $p$ denotes number of vertices in $P$ then $p = mn$ and if $q$ denotes the number of edges in $P$ then $q = (2m - 5)n + (n - 1)$. Also note that the theorem is proved for $m \equiv 0(\text{mod } 2)$ and $m \geq 6$.

The vertices of $P$ are labeled as follows depending on the parameter $n$.

**Labels for the switching vertices $v_i^i$ are given below**

$f(v_1^{2i-1}) = q - \left(\frac{m}{2}\right)(i - 1), \quad \text{for } 1 \leq i \leq \left[\frac{n}{2}\right]$  

$f(v_1^{2i}) = \left(\frac{m}{2}\right)i - 1, \quad \text{for } 1 \leq i \leq \left[\frac{n}{2}\right]$  

**Labels for the vertices $v_{2j+1}$ are given below for $(1 \leq i \leq n), (1 \leq j \leq m)$ are given below**

$f(v_2^{2i-1}) = \left(\frac{m}{2}\right)(i - 1) + (j - 1), \quad \text{for } 1 \leq i \leq \left[\frac{n}{2}\right], 1 \leq j < \frac{m}{2}$  

$f(v_2^{2i}) = (q - 1) - \left(\frac{m}{2}\right)(i - 1) - (j - 1), \quad \text{for } 1 \leq i \leq \left[\frac{n}{2}\right], 1 \leq j < \frac{m}{2}$  

**Case 1: When $n = 1$**
\[ f(v_2^1) = \left\lfloor \frac{q}{2} \right\rfloor \]
\[ f(v_{2j}^1) = \left\lfloor \frac{q}{2} \right\rfloor - 2 - (j - 2) \quad \text{for } 2 \leq j \leq \frac{m}{2} \]

**Case 2: When** \( n = 2 \)
\[ f(v_2^2) = \left\lfloor \frac{q}{2} \right\rfloor \]
\[ f(v_{2j}^2) = \left\lfloor \frac{q}{2} \right\rfloor + (m - 1) + (j - 2) \quad \text{for } 2 \leq j < \frac{m}{2} \]
\[ f(v_{2j}^2) = \left\lfloor \frac{q}{2} \right\rfloor + j \quad \text{for } j = \frac{m}{2} \]

**Case 3: When** \( n > 2 \)

**Case 3.1: When** \( n \equiv 1 \pmod{2} \)
\[ f(v_{2j}^{2i+1}) = \left\lfloor \frac{q}{2} \right\rfloor - i \left( \frac{3m}{2} - 4 \right) - (j - 1) \quad \text{for } 1 \leq i < \left\lfloor \frac{n}{2} \right\rfloor, 1 \leq j < \frac{m}{2} \]
\[ f(v_{2j}^{2i+1}) = \left\lfloor \frac{q}{2} \right\rfloor - i \left( \frac{3m}{2} - 4 \right) + (j - 1) \quad \text{for } 1 \leq i < \left\lfloor \frac{n}{2} \right\rfloor, j = \frac{m}{2} \]

**Case 3.2: When** \( n \equiv 0 \pmod{2} \)
\[ f(v_{2j}^{2i+1}) = \left\lfloor \frac{q}{2} \right\rfloor + \left( \frac{5m}{2} - 6 \right) + (i - 2) \left( \frac{3m}{2} - 4 \right) + (j - 1) \quad \text{for } 2 \leq i < \left\lfloor \frac{n}{2} \right\rfloor, 1 \leq j < \frac{m}{2} \]
\[ f(v_{2j}^{2i+1}) = \left\lfloor \frac{q}{2} \right\rfloor + \left( \frac{5m}{2} - 6 \right) + (i - 2) \left( \frac{3m}{2} - 4 \right) - (j - 1) \quad \text{for } 2 \leq i < \left\lfloor \frac{n}{2} \right\rfloor, j = \frac{m}{2} \]

From the above definition it is clear that all the vertex labels are distinct. The edge labels can be computed from the above vertex labels and they are also found to be distinct from 1 to \( q \). Therefore, the path union of finite copies of vertex switching of cycles is graceful \( m \equiv 0 \pmod{2} \) and \( m \geq 6 \) which is illustrated in Figure 4.

**Illustration:**

Here \( q = 111, n = 7 \) and \( m = 10 \)

**Figure 4: Graceful labeling of path union of vertex switching of cycle** \( C_{10} \)

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4. Conclusion:

In this paper we have proved that the path union of the vertex switching of cycle $C_m$ is graceful where $m \equiv 0 \ (mod\ 2)$, $m \geq 6$. Further we intend to prove the gracefulness of path union of some other graphs.

5. References:


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