A study on the Pseudo-Achromatic Number of Quadrilateral Snakes

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Abstract
The Pseudo-Achromatic number $\Psi_2(G)$ of a graph $G$ is the maximum number of colors which may be assigned to the vertices of $G$ so that for every two colors, there exists adjacent vertices to which these colors are assigned (adjacent vertices may have the same color). This paper derives the general bounds for the Pseudo-Achromatic number of few Snake graphs. An $O(1)$ - approximation algorithm is proposed to determine the pseudo-achromatic number of few quadrilateral Snake graphs.

Key-words: Pseudo-Achromatic number, NP-Completeness, Quadrilateral Snakes.

1. Introduction:
Graph theory offers a path for representing the structure of networks. Graphs are used as device for modelling and description of real world network systems such as transport, water, internet work operations, schemes in the process of production, construction, distribution etc. A graph coloring is one of the oldest and interesting problems that come up in wide applications. Many problems can be formulated as a graph coloring problem including scheduling, time tabling, register allocation etc. A pseudo-coloring of the vertices of a graph $G$ is a coloring assigned to the vertices of $G$ such that adjacent vertices can receive the same color. In this type of coloring, the graph induced by a color class need not be a null graph. A pseudo-complete coloring of a graph $G$ is a pseudo-coloring of the vertices of $G$, such that for any pair of distinct colors, there is at least one edge whose end vertices are colored with this pair of colors. The pseudo-achromatic number $\Psi_2(G)$ of a graph $G$ is the greatest number of colors in a pseudo-complete coloring of $G$. An optimal pseudo-complete coloring is one in which $\Psi_2(G)$ colors are used.

The pseudo-achromatic number of a graph is a natural graph coloring parameters like chromatic and achromatic numbers of a graph $G$. It is easy to see that the pseudo-achromatic number problem is a variation of the graph coloring problem (or the achromatic number problem), [2, 3, 7] the latter pattern requiring the graphs in the partition to be independent sets. [6] Proposed that the pseudo-achromatic problem is NP–Complete even on restricted classes of graph. [2, 4] studied the approximability of the pseudo-achromatic number problem. It was proved in [5] that the problems have a randomized polynomial-time approximation algorithm of ratio $O(\sqrt{\log n})$, which can be de-randomized in polynomial time. This upper bound on the approximation ratio was shown to be asymptotically tight under the randomized model. In [4] researchers have studied on harmonious coloring and labelling of Snake type graphs. In [1] researchers have proved some results on snakes related strongly*- graphs. [8] Revealed the graceful labelling of Quadrilateral Snakes. In this paper
the $O(1)$-approximation algorithm to determine the pseudo-achromatic number of Quadrilateral Snakes is obtained.

2. Main Result:

**Definition 2.1:** A walk is called a path if all its vertices are distinct.

**Remark 2.1:** The path graph $P_n$ has $n$ vertices and $n - 1$ edges.

**Remark 2.2:** The two end vertices of $P_n$ are degree one and $n - 2$ internal vertices are of degree two.

**Definition 2.2:** A complete graph with $n$ vertices (denoted by $K_n$) is a graph with $n$ vertices in which each vertex is connected to each of the others (with one edge between each pair of vertices).

**Definition 2.3:** A cycle of length $n$ is the graph $C_n$ on $n$ vertices $\{v_0, v_1, \ldots, v_{n-1}\}$ with $n$ edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_0)$.

**Definition 2.4:** A subgraph $H$, spans a graph $G$ and is a spanning subgraph, or factor of $G$, if it has the same vertex set as $G$.

**Definition 2.5:** A quadrilateral snake $Q_n$ is obtained from a path $P_n$ by replacing each edge of the $P_n$ by a cycle $C_4$.

**Definition 2.6:** An alternate quadrilateral snake $A(Q_n)$ is obtained from a path $P_n$ by replacing each alternate edge of the $P_n$ by a cycle $C_4$.

**Definition 2.7:** A double quadrilateral snake $D(Q_n)$ is consisting of two quadrilateral snakes that have common path.

**Definition 2.8:** An alternate double quadrilateral snake $A[D(Q_n)]$ is consisting of two alternate quadrilateral snakes that have common path.

3. PSEUDO-ACHROMATIC NUMBER FOR QUADRILATERAL SNAKE GRAPH $Q_n$:

Let $P_n$ be a path graph with $n$ vertices $n - 1$ edges and $C_4$ be a complete graph with 4 vertices. The Quadrilateral snake graph $Q_n$ is obtained from a path $P_n$ by replacing each edge of the $P_n$ by a cycle $C_4$. See Figure 1.

**Figure 1:** Quadrilateral Snake $Q_n$  **Figure 2:** Pseudo-Achromatic labelling of Quadrilateral Snake $Q_n$

Let $V(Q_n)$ and $E(Q_n)$ be the vertex set and edge set of $Q_n$ respectively. Now $V(Q_n) = \{(v_i, v_{i+1}), 1 \leq i \leq n\} \cup \{u_i, u_{i+1}, 1 \leq i \leq n\}$ and $E(Q_n) = \{v_i, v_{i+1}; 1 \leq i \leq n\} \cup \{v_i; 1 \leq i \leq n\} \cup \{u_i, u_{i+1}; 1 \leq i \leq n\} \cup \{u_i, v_{i+1}; 1 \leq i \leq n\}$ $|V(Q_n)| = 3n - 2$ and $|E(Q_n)| = 4(n - 1)$. Maximum degree $\Delta[Q_n] = 4$, Minimum degree $\delta[Q_n] = 2$. 
Algorithm Pseudo-Achromatic for Quadrilateral Snake $Q_n$:

**Theorem 3.1:** For $n \geq 1$, $\psi_s(Q_n) \geq 2k + 2$.

Let $k$ be an integer such that $\frac{k(k+1)}{2} \leq t \leq \frac{(k+1)(k+2)}{2}$. Partition the set of $tC_4's$ into $D_k \cup D_{k-1} \cup ... \cup D_1$.

For $0 \leq j \leq k - 1$ label $D_{k-j}$ as follows:

Label the bottom vertices of each $C_4$ in $D_{k-j}$ as $2j + 1$ and $2j + 2$. Label the top vertices as $2j + 3, 2j + 4, ..., (2j + 2(k - j) + 1)$ respectively, if the number of $C_4's$ in $D_{k-j}$ is even and label the top vertices as $2j + 3, 2j + 4, ..., 2j + (2(k - j) + 2)$ respectively, if the number of $C_4's$ in $D_{k-j}$ is odd, in clockwise direction. In turn the top vertices of last $C_4$ in every $D_{k-j}$ receive the same label. Thus the labelling of these vertices gives a Pseudo-Achromatic labelling of $Q_n$ and hence $\psi_s(Q_n) \geq 2k + 2$. This completes the proof.

**Proof of correctness:** As the bottom vertices labelled $2j + 1$ and $2j + 2$ are adjacent in $D_{k-j}$ they in turn are adjacent to top vertices labelled $2j + 3, 2j + 4, ..., 2j + 2(k - j)$ or $2j + 3, 2j + 4, ..., 2j + (2(k - j) + 1)$ and $2j + 3, 2j + 4, ..., 2j + 2(k - j)$ or $2j + 3, 2j + 4, ..., 2j + (2(k - j) + 2)$ as the number of $C_4's$ in $D_{k-j}$ is even or odd respectively. Since the top vertices of last $C_4$ in every $D_{k-j}$ receive the same label, the labelling is pseudo. Therefore the vertex induces the pseudo-achromatic labelling yielding pseudo-achromatic number $\frac{k(k+1)}{2}C_4$ for $D_k$ to be atleast $2k + 2$. See figure 2.

The following theorem is straightforward as the number of edges of $Q_n$ is $4(n - 1)$.

**Theorem 3.2:** Let $Q_n$ be a Quadrilateral Snake graph of dimension of $r, r \geq 2$. Then $\psi_s(Q_n) \leq -\frac{1+\sqrt{32n-31}}{2}.

Theorem 3.1 and Theorem 3.2 imply the following result.

**Theorem 3.3:** Let $Q_n$ be a Quadrilateral Snake graph of dimension of $r, r \geq 2$. Then $\frac{4}{5} \left( -\frac{1+\sqrt{32n-31}}{2} \right) \leq \psi_s(Q_n) \leq -\frac{1+\sqrt{32n-31}}{2}$.

Since $Q_n$ have $3n - 2$ vertices we have the following result.

**Theorem 3.4:** There is an $O(1)$- approximation algorithm to determine the pseudo-achromatic number of $Q_n$ on $3n - 2$ vertices.

**Proof:** A lower bound for the pseudo-achromatic number of $Q_n$ is $\left( -\frac{1+\sqrt{32n-31}}{2} \right)$ and the expected pseudo-achromatic number for $Q_n$ is $\left( -\frac{1+\sqrt{32n-31}}{2} \right)$. The ratio of the two numbers is of $O(1)$ and hence there is an $O(1)$- approximation algorithm to determine the pseudo-achromatic number of $Q_n$.

4. **PSEUDO-ACHROMATIC NUMBER FOR ALTERNATE QUADRILATERAL SNAKE A(Qn):**

Let $P_n$ be a path graph with $n$ vertices $n - 1$ edges and $C_4$ be a complete graph with four vertices. The alternate Quadrilateral snake graph $A(Q_n)$ is obtained from a path $P_n$ by replacing each alternate edge of the $P_n$ by a cycle $C_4$. See Figure 3.
Let $V(A(Q_n))$ and $A(Q_n)$ be the vertex set and edge set of $A(Q_n)$ respectively. Now

$V(A(Q_n)) = \{(v_i, v_{i+1}); 1 \leq i \leq n\} \cup \{u_i, u_{i+1}; 1 \leq i \leq n\}$

$E(A(Q_n)) = \{v_i v_{i+1}; 1 \leq i \leq n\} \cup \{v_i u_j; 1 \leq i \leq n\} \cup \{u_i u_{i+1}; 1 \leq i \leq n\}$

and $\{u_{i+1} v_{i+2}; 1 \leq i \leq n\}$ connected by a path $\{v_i v_{i+2}; 1 \leq i \leq n\}$.

Note that $|V(A(Q_n))| = \begin{cases} \frac{2n}{2}, & \text{if } n \text{ is even} \\ \frac{2n - 1}{2}, & \text{if } n \text{ is odd} \end{cases}$ and

$|E(A(Q_n))| = \begin{cases} \frac{5n}{2} - 1, & \text{if } n \text{ is even} \\ \frac{5(n - 1)}{2}, & \text{if } n \text{ is odd} \end{cases}$.

Maximum degree $\Delta[A(Q_n)] = 3$, Minimum degree $\delta[A(Q_n)] = 2$.

**Algorithm Pseudo-Achromatic for Alternate Quadrilateral Snake $A(Q_n)$:**

**Theorem 4.1:** For $n \geq 1$, $\psi_s(A(Q_n)) \geq 2k + 1$.

Let $k$ be an integer such that $\frac{k(k+1)}{2} \leq t \leq \frac{(k+1)(k+2)}{2}$. Partition the set of $tC_4$’s into $D_k \cup D_{k-1} \cup ... \cup D_1$.

For $0 \leq j \leq k - 1$ label $D_{k-j}$ as follows:

Label the bottom vertices of each $C_4$ in $D_{k-j}$ as $2j + 1$ and $2j + 2$. Label the top vertices as $2j + 3, 2j + 4, ... , 2j + 2(k-j) + 2$ and $2j + 3, 2j + 4, ... , 2j + 2(k-j) + 1$ respectively, in the alternate $C_4$’s of $D_{k-j}$ in clockwise direction. In turn the top vertices of last $C_4$ in every $D_{k-j}$ receive the same label, if the number of $A(Q_n)$ is odd. Thus the labelling of these vertices gives a Pseudo-Achromatic labelling of $A(Q_n)$ and hence $\psi_s(A(Q_n)) \geq 2k + 2$. This completes the proof.

**Proof of correctness:** As the bottom vertices labelled $2j + 1$ and $2j + 2$ are adjacent in $D_{k-j}$ they in turn are adjacent to top vertices labelled $2j + 3, 2j + 4, ... , 2j + 2(k-j) + 2$ or $2j + 3, 2j + 4, ... , 2j + 2(k-j) + 1$ and as the number of $C_4$’s in $D_{k-j}$ is even or odd respectively. Since the top vertices of last $C_4$ in every $D_{k-j}$ receive the same label in the odd number of $A(Q_n)$, the labelling is pseudo. Therefore the vertex induces the pseudo-achromatic labelling yielding pseudo-achromatic number $\frac{k(k+1)}{2}C_4$ for $D_k$ to be atleast $2k + 2$. See figure 4.

The following theorem is straightforward as the number of edges of $A(Q_n)$ is $\begin{cases} \frac{5n}{2} - 1, & \text{if } n \text{ is even} \\ \frac{5(n - 1)}{2}, & \text{if } n \text{ is odd} \end{cases}$. 

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**Figure 3:** Alternate Quadrilateral Snake $A(Q_n)$

**Figure 4:** Pseudo-Achromatic labelling of Alternate Quadrilateral Snake $A(Q_n)$
Theorem 4.2: Let \( A(Q_n) \) be a Quadrilateral Snake graph of dimension of \( r, r \geq 2 \). Then \( \psi_s(A(Q_n)) \leq \left\{ \begin{array}{ll} \frac{-1 + \sqrt{20n - 7}}{2}, & \text{if } n \text{ is even} \\ \frac{-1 + \sqrt{40n - 39}}{2}, & \text{if } n \text{ is odd} \end{array} \right. \).

Theorem 4.3: Let \( A(Q_n) \) be a Quadrilateral Snake graph of dimension of \( r, r \geq 2 \). Then \( \left\{ \begin{array}{ll} \frac{-1 + \sqrt{20n - 7}}{2}, & \text{if } n \text{ is even} \\ \frac{-1 + \sqrt{40n - 39}}{2}, & \text{if } n \text{ is odd} \end{array} \right. \) \leq \psi_s(A(Q_n)) \leq \left\{ \begin{array}{ll} \frac{-1 + \sqrt{20n - 7}}{2}, & \text{if } n \text{ is even} \\ \frac{-1 + \sqrt{40n - 39}}{2}, & \text{if } n \text{ is odd} \end{array} \right. \).

Since \( A(Q_n) \) have \( \left\{ \begin{array}{ll} 2n, & \text{if } n \text{ is even} \\ 2n - 1, & \text{if } n \text{ is odd} \end{array} \right. \) vertices we have the following result.

Theorem 4.4: There is an \( O(1) \)-approximation algorithm to determine the pseudo-achromatic number of \( A(Q_n) \) on \( n \) vertices.

Proof: A lower bound for the pseudo-achromatic number of \( A(Q_n) \) is \( \left\{ \begin{array}{ll} \frac{-1 + \sqrt{20n - 7}}{2}, & \text{if } n \text{ is even} \\ \frac{-1 + \sqrt{40n - 39}}{2}, & \text{if } n \text{ is odd} \end{array} \right. \) and the expected pseudo-achromatic number for \( A(Q_n) \) is \( \left\{ \begin{array}{ll} \frac{-1 + \sqrt{20n - 7}}{2}, & \text{if } n \text{ is even} \\ \frac{-1 + \sqrt{40n - 39}}{2}, & \text{if } n \text{ is odd} \end{array} \right. \). The ratio of the two numbers is of \( O(1) \) and hence there is an \( O(1) \)-approximation algorithm to determine the pseudo-achromatic number of \( A(Q_n) \).

5. PSEUDO-ACHROMATIC NUMBER FOR DOUBLE QUADRILATERAL SNAKE GRAPHS \( D(Q_n) \):

Let \( P_n \) be a path graph with \( n \) vertices \( n - 1 \) edges and \( C_4 \) be a complete graph with four vertices. A double triangular snake \( D(Q_n) \) consists of two Quadrilateral snakes that have a common path. See Figure 5.

![Figure 5](image1)

![Figure 6](image2)

**Figure 5:** Double Quadrilateral Snake \( D(Q_n) \).  **Figure 6:** Pseudo-Achromatic labelling of Double Quadrilateral Snake \( D(Q_n) \).

Let \( V[D(Q_n)] \) and \( E[D(Q_n)] \) be the vertex set and edge set of \( D(Q_n) \) respectively. Now \( V[D(Q_n)] = \{(v_i, v_{i+1}), 1 \leq i \leq n\} \cup \{(u_i, u_{i+1}), 1 \leq i \leq n\} \cup \{(w_i, w_{i+1}), 1 \leq i \leq n\} \)

\( E[D(Q_n)] = \{(v_i, v_{i+1}), 1 \leq i \leq n\} \cup \{u_i, u_{i+1}, 1 \leq i \leq n\} \cup \{w_i, w_{i+1}, 1 \leq i \leq n\} \cup \{v_i, w_{i+1}, 1 \leq i \leq n\} \cup \{v_{i+1}, w_i, 1 \leq i \leq n\} \cup \{w_{i+1}, v_i, 1 \leq i \leq n\} \cup \{v_{i+1}, w_{i+1}, 1 \leq i \leq n\} \cup \{w_i, v_{i+1}, 1 \leq i \leq n\} \)

\(|V[D(Q_n)]| = 5n - 4 \text{ and } |E[D(Q_n)]| = 7(n - 1)\).

Maximum degree \( \Delta[D(Q_n)] = 6 \), Minimum degree \( \delta[D(Q_n)] = 2 \).
Algorithm Pseudo-Achromatic for Double Quadrilateral Snake Graph $D(Q_n)$:

**Theorem 5.1**: For $n \geq 1$, $\psi_s(D(Q_n)) \geq 2k + 2$.

Let $k$ be an integer such that $\frac{k(k+1)}{2} \leq t \leq \frac{(k+1)(k+2)}{2}$. Partition the set of $tC_4's$ into $D_k \cup D_{k-1} \cup ... \cup D_1$.

For $0 \leq j \leq k - 1$ label $D_{k-j}$ as follows:

Label the middle vertices of each $C_4$ in $D_{k-j}$ as $2j + 1$ and $2j + 2$. Label the top and bottom vertices in reverse order as $2j + 3, 2j + 4, ..., 2j + (2(k-j) + 2)$ alternately in each $C_4$ of every $D_{k-j}$, in clockwise direction and hence $\psi_s(D(Q_n)) \geq 2k + 2$. This completes the proof.

**Proof of correctness**: As the middle vertices labelled $2j + 1$ and $2j + 2$ are adjacent in $D_{k-j}$ they in turn are adjacent to top and bottom vertices labelled alternately as $2j + 3, 2j + 4, ..., 2j + (2(k-j) + 2)$ or $2j + 3, 2j + 4, ..., 2j + (2(k-j) + 2)$. Therefore the vertex induces the pseudo-achromatic labelling yielding pseudo-achromatic number $\frac{k(k+1)}{2}C_4$ for $D_k$ to be at least $2k + 2$. See figure 6.

The following theorem is straightforward as the number of edges of $D(Q_n)$ is $7(n-1)$.

**Theorem 5.2**: Let $D(Q_n)$ be a Double Quadrilateral Snake graph of dimension of $r, r \geq 2$. Then $\psi_s(D(Q_n)) \leq \frac{-1 + \sqrt{28n-27}}{2}$.

Theorem 5.1 and Theorem 5.2 imply the following result.

**Theorem 5.3**: Let $D(Q_n)$ be a Double Quadrilateral Snake graph of dimension of $r, r \geq 2$. Then $\left[\frac{-1 + \sqrt{28n-27}}{2}\right] \leq \psi_s(D(Q_n)) \leq \frac{-1 + \sqrt{28n-27}}{2}$. Since $D(Q_n)$ have $5n - 4$ vertices we have the following result.

**Theorem 5.4**: There is an $O(1)$-approximation algorithm to determine the pseudo-achromatic number of $D(Q_n)$ on $n$ vertices.

**Proof**: A lower bound for the pseudo-achromatic number of $D(Q_n)$ is $\left[\frac{-1 + \sqrt{28n-27}}{2}\right]$ and the expected pseudo-achromatic number for $D(Q_n)$ is $\frac{-1 + \sqrt{28n-27}}{2}$. The ratio of the two numbers is of $O(1)$ and hence there is an $O(1)$-approximation algorithm to determine the pseudo-achromatic number of $D(Q_n)$.

6. PSEUDO-ACHROMATIC NUMBER FOR ALTERNATE DOUBLE QUADRILATERAL SNAKE GRAPH $DA(Q_n)$:

Let $P_n$ be a path graph with $n$ vertices $n-1$ edges and $C_4$ be a complete graph with four vertices. An alternate double triangular snake $DA(Q_n)$ consists of two alternate double quadrilateral snakes that have a common path. See Figure 7.

Figure 7: Alternate Double Quadrilateral Snake $DA(Q_n)$.

Figure 8: Pseudo-Achromatic labelling of Snake $DA(Q_n)$.
Let $V[DA(Q_n)]$ and $E[DA(Q_n)]$ be the vertex set and edge set of $DA(Q_n)$ respectively. Now $V[DA(Q_n)] = \{(v_i, v_{i+1}) : 1 \leq i \leq n\} \cup \{(u_i, u_{i+1}) : 1 \leq i \leq n\} \cup \{(w_i, w_{i+1}) : 1 \leq i \leq n\}$ and $E[DA(Q_n)] = \{(v_i, v_{i+1}) : 1 \leq i \leq n\} \cup \{(u_i, u_{i+1}) : 1 \leq i \leq n\} \cup \{(w_i, w_{i+1}) : 1 \leq i \leq n\}$, connected by a path $\{(v_{i+1}, v_{i+2}) : 1 \leq i \leq n\}$.

$|V[DA(Q_n)]| = \begin{cases} 3n, & \text{if } n \text{ is even} \\ 3n - 2, & \text{if } n \text{ is odd} \end{cases}$ and $|E[DA(Q_n)]| = \begin{cases} 4n - 5, & \text{if } n \text{ is even} \\ 4(n - 1), & \text{if } n \text{ is odd} \end{cases}$.

Maximum degree $\Delta[DA(Q_n)] = 4$, Minimum degree $\delta[DA(Q_n)] = 2$.

Algorithm Pseudo-Achromatic for Alternate Double Quadrilateral Snake Graph $DA(Q_n)$:

**Theorem 6.1:** For $n \geq 1$, $\psi_s(DA(Q_n)) \geq 2k + 2$.

Let $k$ be an integer such that $\frac{k(k+1)}{2} \leq t \leq \frac{(k+1)(k+2)}{2}$. Partition the set of $tC_4$s into $D_k \cup D_{k-1} \cup \ldots \cup D_1$.

For $0 \leq j \leq k - 1$ label $D_{k-j}$ as follows:

Label the middle vertices of each $C_4$ in $D_{k-j}$ as $2j + 1$ and $2j + 2$. Label the top vertices and bottom vertices in reverse order as $2j + 3, 2j + 4, \ldots, 2j + (2(k - j) + 2)$ alternately in each $C_4$s of every $D_{k-j}$ in clockwise direction. The path between each $C_4$ is labeled as $2j + 2$ and $2j + 1$. Thus the labelling of these vertices gives a Pseudo-Achromatic labelling of $DA(Q_n)$ and hence $\psi_s(DA(Q_n)) \geq 2k + 2$. This completes the proof.

**Proof of correctness:** As the middle vertices labelled $2j + 1$ and $2j + 2$ are adjacent in $D_{k-j}$ they in turn are adjacent to top and bottom vertices labelled alternately as $2j + 3, 2j + 4, \ldots, 2j + (2(k - j))$ or $2j + 3, 2j + 4, \ldots, 2j + (2(k - j) + 2)$. Therefore the vertex induces the pseudo-achromatic labelling yielding pseudo-achromatic number $\frac{k(k+1)}{2}C_4$ for $D_k$ to be atleast $2k + 2$. See figure 8.

The following theorem is straight forward as the number of edges of $DA(Q_n)$ is $\begin{cases} 4n - 5, & \text{if } n \text{ is even} \\ 4(n - 1), & \text{if } n \text{ is odd} \end{cases}$.

**Theorem 6.2:** Let $DA(Q_n)$ be an Alternate Double Quadrilateral Snake graph of dimension of $r, r \geq 2$. Then $\psi_s(DA(Q_n)) \leq \begin{cases} -1 + \sqrt{32n - 39} \div 2, & \text{if } n \text{ is even} \\ -1 + \sqrt{32n - 31} \div 2, & \text{if } n \text{ is odd} \end{cases}$.

Theorem 6.1 and Theorem 6.2 imply the following result.

**Theorem 6.3:** Let $D(Q_n)$ be a double Quadrilateral Snake graph of dimension of $r, r \geq 2$.

Then $\begin{cases} -1 + \sqrt{32n - 39} \div 2, & \text{if } n \text{ is even} \\ -1 + \sqrt{32n - 31} \div 2, & \text{if } n \text{ is odd} \end{cases} \leq \psi_s(DQ_n) \leq \begin{cases} -1 + \sqrt{32n - 39} \div 2, & \text{if } n \text{ is even} \\ -1 + \sqrt{32n - 31} \div 2, & \text{if } n \text{ is odd} \end{cases}$.

Since $DA(Q_n)$ have $\begin{cases} 3n, & \text{if } n \text{ is even} \\ 3n - 2, & \text{if } n \text{ is odd} \end{cases}$ vertices we have the following result.

**Theorem 6.4:** There is an $O(1)$-approximation algorithm to determine the pseudo-achromatic number of $DA(Q_n)$ on $n$ vertices.
Proof: A lower bound for the pseudo-achromatic number of $DA(Q_n)$ is

\[
\left\{ \begin{array}{ll}
-1 + \frac{\sqrt{52n-38}}{2}, & \text{if } n \text{ is even} \\
-1 + \frac{\sqrt{52n-31}}{2}, & \text{if } n \text{ is odd}
\end{array} \right.
\]

and the expected pseudo-achromatic number for $DA(Q_n)$ is

\[
\left\{ \begin{array}{ll}
-1 + \frac{\sqrt{52n-38}}{2}, & \text{if } n \text{ is even} \\
-1 + \frac{\sqrt{52n-31}}{2}, & \text{if } n \text{ is odd}
\end{array} \right.
\]

The ratio of the two numbers is of $\mathcal{O}(1)$ and hence there is an $\mathcal{O}(1)$ - approximation algorithm to determine the pseudo-achromatic number of $DA(Q_n)$.

CONCLUSION

In this paper an $\mathcal{O}(1)$ - approximation algorithm to determine the pseudo-achromatic number of few Quadrilateral Snake graphs is discussed. Finding proficient approximation algorithms to determine pseudo-achromatic number for other derived networks is quite inspiring.

References

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